Spatial Structure of Wages and Rents

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Abstract

Urban economic literature usually focuses on Central Business District model, which explains rental gradient around city center and is based on assumption that agents are dispersed in space, but all work in one point. The basic idea of the present model is that job exist in all points of continuous space, but high wages are offered only in some set of points. It can be either due to superior technology (hi-tech center) or difference in tax laws (Monaco) or just wage differential between rural and urban area. Since housing supply is inelastic in the short run, the immediate response to such situation is travelling to work where wages are higher from the location where housing is cheaper. A short-run equilibrium presents such wage distribution and rental price for housing that nobody can make spatial arbitrage. The long run dynamics includes both investment in housing as well as labor migration. Industrialization first has pushed migration to cities, the places of superior technology, but later some people returned to the city neigbourhoods, which were offering cheaper housing and possibility to travel to work to city. This pattern of migration is named peri-urbanization and was typical for Spanish internal mobility in 1980-ies.

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1 Introduction

The spatial structure plays a crucial role for the models of land rent and pricing real estate. The prices of physically identical housing differ in different parts of the same city, as well as across regions and countries. This difference can be studied both at empirical and theoretical levels. While the price patterns for real estate are represented graphically on maps, the spatial difference of wage pattern is not so well documented. The technical problem is that non-spatial job heterogeneity is much wider than heterogeneity of real estate. Nevertheless, it is a stylized fact that the City of London pays higher wage on average than Greater London, while London area is richer than UK on average. Similar situation takes place in other big agglomerations, like Moscow, Paris or Madrid. On one hand, agglomerations provide higher income, but on the other hand, expenditures on real estate are also higher there. This had to deal with location rent, which results from local competition for non-tradable goods among richer set of consumers.

The modern theory of urban rent is based on microeconomic theory of land rent developed in the works of Alonso [1], Fujita [2] and many other authors ([3] is a good collection of works in the area). The goal of the present paper is to take into account both rent and wage heterogeneity across space. This difference is very pronounced at both macro (differences in wage for identical skill and prices for identical flat across countries and regions) and micro (the same differences inside one city, between city and rural neighbourhood) levels.

International trade theory usually splits goods into tradable and non-tradable. However, in our increasingly integrated and "seamless" (in the sense of Krugman) world many consumers can engage in possibilities of spatial arbitrage. People do not necessarily work in the same place where they live. This brings an interlink between closely located labor and real estate markets. Especially the effect becomes pronounced near the border of economies with different levels of GDP per capita. American industry has created a lot of jobs on the border with Mexico, both to fight illegal migration and to take advantage of cheaper labor [6, 7]. Since there was no infrastructure there (even water is scarce good), few people settled there, but many travel to work on the border from the neighbouring areas. This should push wages up in the neighbouring areas, but the observed wages decline fast with

the distance from the border. Another example is related to Monaco. This small country is a paradise for rich people, who can enjoy paying low taxes if they reside there. The competition across them for limited quantity of housing push the prices up to such a level that middle or low income class is not able to reside there. However, workers from neighouring areas if France can enjoy spatial arbitrage, living in reasonably priced houses just on the border with Monaco and working in Monaco for higher salaries.

The possibility of spatial arbitrage is limited in space. It depends on transport cost. With zero transport cost and no legal barriers, wages would equalize in the world. However, this does not happen. Even the differences across regions inside European community, which has established the policy of free labor mobility, do not vanish too fast. While it is intuitively clear that costly spatial arbitrage is responsible for the existing spatial patterns of wages and prices for real estate, I am not aware of any serious fundamental research oriented on building a theoretical model of such phenomena.[1] Few simple models that are presented in this paper attempt to fill this gap.

On individual level, there are two instruments to use spatial arbitrage. The first is migration. The literature, both empirical and theoretical, on migration, is wide. It makes sense to focus here only on some part of it which gives stylized facts useful for understanding local spatial structure of wages and rents. Rental and wage differentials across different points in space is a driving force for migration of people but it is also affected by it. Migration flows can be across countries, regions of a country or within a region. The last type of migration forms the main share of mobility inside many European countries in recent years. For example, in Italy in 1986-87 intraregional migration (one within a region) was 830 000, or 72 % of overall migration [11].

During the period 1960-80 Spain was experiencing a process of urbanization, driven by 40 % difference between net earnings between industry and

[1] I have to admit that literature in urban economics did a lot in explaining rent structure in cities. However, it normally assumed one location of employment and commuting with this center. There exists research about agglomerations with several centers, used to explain the spatial patterns of agglomerations in Japan, for example. It is necessary to mention the fundamental work of Christaller [8], where all space is decomposed into hexagon structures of different order. Both directions, however, are aside from the approach of this paper.

agriculture [5]. However, during the next decade the process became partly reversed. Big industrial areas like Madrid and Barcelona, previously attracting centers for migrants, started to repell them and caused the phenomenon of periurbanization. The improvement of urban transportation infrastructure allowed some agents to have residences in about one-hour journey away from their work. This movement was especially significant for young couples with low and medium income, who could not afford a flat in the city center. But some agents with higher income bought housing in periurban rings either to benefit from larger affordable size of it or to seek a better quality of life in rural villages [5]. The process of periurbanization is a representation of another mechanism of spatial arbitrage: people do not necessarily live and work in the same place. They may commute to the place of their work. Moreover, the place of work may have many spatial locations and people do not necessarily work in the city center like it is usually considered in models of urban economists.

Similar process was taking place near Greater London. As shown in [12], between 1983-92 the main interregional migration flows in UK (those of size above 100 000) have occured to periurban ring of London. While residing in the City of London is quite expensive and most people commute to work in the Center from non-central parts of Greater London, the average price of housing in the whole Greater London is currently 178 th. pounds, which is significantly above the average of England and Wales (106.5 th. pounds)[2]. The described periurbanization process has pushed prices up in South-East Anglia to 137 th. pounds. Moreover, the annual increase over the last year was higher in East Anglia (17 %) which is periurban ring of Greater London than in Greater London itself (13 %) or England and Wales average (10 %).

The examples above show that spatial structure of wages and rents is indeed an important process for agents' decision on spatial mobility. On the other hand, this mobility gives a backward link causing changes in rents and wages over time. The full dynamic model of the process seems to be quite a complex task, but some analytical explanation of periurbanization process is quite desirable.

[2]The data are from www.landreg.gov.uk and were represented in the article of Christoffer Adams published in the "Financial Times", 26 February 2001.

The main problem of linking migration theory with classical microeconomic models is that despite huge migration flows wage equalization across different regions and countries of the world is not observed. Usually, migration models use some frictions, like moving costs or uncertainty to find a job. It is worthwhile mentioning, that high cost of housing in rich areas works as an instrument preventing over-migration there. Rental price is an important factor to make people in different locations indifferent in their decision to move.

The second possibility of spatial arbitrage by individuals is to live in one place and work in other place. People, travelling to their work, pay transport cost, which is an increasing (usually linear) function of distance. This possibility is in the core of central business district (CBD) models. In the simplest version of CBD model, where the jobs are offered only in the city center, agents who live away from the center, pay land rent, just equal to the explicit and implicit cost of going to work in center.

Normally, jobs are offered not only in city centers, but in all locations. When city just starts, it attracts peasants by higher wages. This is the process of urbanization. At the later stages, we observe some people who continue to reside in rural area but choose to work in a city. Finally, the process of periurbanization, or moving away from the city center to closer neighbourhood, can take place. It was empirically observed as the typical pattern of Spanish internal migration in 1980-ies [5].

Goals of the paper. The first goal is to demonstrate the effect of arbitrage possibilities of workers in space and to see what kind of equilibrium wage profile can be created. The second goal is to demonstrate the common effect of spatial labor mobility and market for real estate in establishing equilibrium spatial patterns of house pricing and wages as well as commuting flows of some workers who prefer to live and work in different locations. This models can be used for explanation of the effect of periurbanization which was observed as the dominant effect of spatial labor mobility in Spain in 1980-ies, and takes place in other countries as well.

Structure of the paper. The paper starts from a sequence of simple models which describe an emergence of equilibrium spatial structures. The

driving force for their emergence is exogeneous spatial heterogeneity of labor demand functions. The possibilities of workers to commute between residences and work tend to equalize wages across space while housing rent tend to diminish locational advantage. The complex spatial pattern of wages and housing rents emerges even in the case of agents with equal wealth. Next sections deal with extensions which include a possibility to subrent part of house and the dynamics of construction. Later agents are assumed to be heterogeneous in their wealth, but the housing product offered by market is homogeneous in physical properties and heterogeneous in location. Finally, conclusions and policy implications are formulated.

2 Assumptions of the Model

2.1 Two Spatial Structures

There are two spatial structures. Both use continuous two-dimensional Euclidean space. The first, with linear symmetry, typically describes the situation close to a border between a rich and a poor economy. The rich economy which is located at all points of left semiplane, offers high wage w_A. The poor economy is located at right semiplane $x > 0, y \in R$ and offers low wage w_B. Economy is symmetric with respect to the second spatial variable y. Hence, the only relevant spatial parameter is the distance to the border, r, which equals to x. The labor supply in rich economy is assumed to be perfectly elastic. This makes wage in country A spatially homogeneous. The main concern would e about wage structure in country B as a function of the distance to the border r.

The second has a radial symmetry. It can be used to model the effects of urbanization and periurbanization. In this specification, the CBD offers a high wage, W, which does not depend on labor supply (perfectly elastic labor demand). The rural area offers low wage, w, which may depend on different factors, which are to be discussed later.

2.2 Types of Agents and Their Preferences

1. Agents are identical in their preferences, which are assumed to be Cobb-Douglas in consumption and leisure. But they are initially endowed with a

house in different location, and this makes them heterogeneous. If an agent resides in location r (measured as distance from the border or CBD) and works in location $\bar{r}$, then he obtains utility

$$U = (w(\bar{r}) - P(r) - 2t|r - \bar{r}|)^{\alpha}(l_0 - 2|r - \bar{r}|/V)^{1-\alpha}, \tag{1}$$

where $R(r)$ is housing price including spatial rent, t - unit distance transport cost, V - the speed of transportation, l_0 - daily gross leisure endowment (including transportation time but excluding time for work and basic needs), $0 < \alpha < 1$ is a parameter.

2. Different technologies have different locations, and thus wage structure has a spatial heterogeneity. It is assumed that the central ($R = 0$) or border ($r = 0$) location can supply any amount of jobs at the same wage $w(0)$ while in any of other locations there is local monopsony which can attract workers by paying not their marginal value of the product but opportunity cost.

3. In different sub-models of the paper housing markets can be closed of opened. Later agents may also differ in wealth endowment.

3 A Set of Simple Models

In this section agents are assumed to be homogeneous. That means that they should obtain the same utility in all locations. We start with a simple model with business center or border which assumes no housing market, but initial endowment with non-tradable house in some location. Thus, $P(r) = 0$ in (1). Then it is shown that model with equilibrium wage around business center (CBD model) formally has the same solution. This allows to make no formal distinction whether we consider a border of CBD model in later elaborations of the next section. Then only housing market is considered in the assumption that job can be offered at the point zero (either border, or business center). Finally, two models are linked together. This allows to obtain a simultaneous pattern of spatial structures of wages and rents.

3.1 A Simple CBD Model

The spatial structure of this model has a radial symmetry, and polar coordinates are more useful here. Radial symmetry keeps only one variable, the distance R to the business center, located at $R = 0$, as important. The business center offers a perfectly elastic supply of jobs with wage $w(0)$. Agents are endowed with non-tradable housing in different distances R. There exists a minimal wage w_a, which is significantly less than $w(0)$, and can be thought as benefits derived by farming if agents choose to be self-employed. In each location there is also a small firm, assumed to be a monopsonist and paying the opportunity cost to workers. In the absence of business center it will pay just w_a, but as business center emerges, it starts paying different wages in different locations. An agent residing at R is indifferent between working at R or at business center $R = 0$ if the following condition holds:

$$w(R)^{\alpha} l_0^{1-\alpha} = (w(0) - 2tR)^{\alpha}(l_0 - \frac{2R}{V})^{1-\alpha}. \tag{2}$$

Taking into account the opportunity cost and simplifying formulae, the spatial wage structure is given by:

$$w(R) = \max(w^*(R), w_a), \tag{3}$$

$$w^*(R) = (w(0) - 2tr)(l_0 - \frac{2r}{V})^{\frac{1}{\alpha}-1}. \tag{4}$$

3.2 A Simple Model with a Border

The goal of this model is to study the wage dynamics in low economy after opening the border.[3] It is assumed that the supply of living space in houses is fixed in the short run, since construction takes time. The construction of the solution is done in several steps. First, only arbitrage possibility between the border and house are considered.

No arbitrage condition between working where one lives and on the border. A marginal worker, living at the distance r from the border, is indifferent between moving to work to the border every day to get wage

[3]Mathematically, the process of urbanization is described by the same formulae (with the only difference, that R, the distance to CBD is replaced by r, the distance from the border, in this case.

w_0 and working at the place, where he lives, for wage $w(r)$, when the spatial wage structure satisfies the following requirement:

$$w(r) = \max(w^*(r), w_B), \tag{5}$$

$$w^*(r) = (w(0) - 2tr)(l_0 - \frac{2r}{V})^{\frac{1}{\alpha}-1}. \tag{6}$$

This formula describes a continuous function, decreasing from $r = 0$ to $r = r^*$, where r^* is the point, after which no movement occurs and the wage is equal to $w(r) = w(r^*) = w_B$, for all $r > r^*$.

The generalizations can go in different directions. At the first stage it is useful to forget about wage structure (assuming no jobs at other locations except zero) but to open housing markets. This gives a simple CBD model of the type well known to urban economists. Its was mentioned in [4] and differs from usual urban models by not considering a possibility to select a size of housing and considering explicitly the leisure loss from travel.

3.3 Rent Structure around CBD

The agents are indifferent between residing in all locations and paying housing rent $P(R)$ if all of them work at the business center $R = 0$ for the wage $w(0)$, paying direct transport cost for daily round trip $2tR$ and indirectly via leisure loss by transportation time $2R/V$ if the following condition holds:

$$(w(0) - P(0))^{\alpha} l_0^{1-\alpha} = (w(0) - 2tR)^{\alpha}(l_0 - \frac{2R}{V})^{1-\alpha}. \tag{7}$$

This gives a rent structure around CBD:

$$P(R) = w(0) - 2tR - \frac{(w(0) - P(0))}{(1 - 2R/(l_0 V))^{(1-\alpha)/\alpha}}. \tag{8}$$

Note that $P' < 0$ and $P'' < 0$, which means that price gradient becomes steeper with an increase in distance. Similar formula should hold for $R(r)$ in the model with the distance from a border.

3.4 Wage and Rent Spatial Structure

The goal is to put together spatial structures for wages and house rents obtained in the previous subsections. For simplicity, consider the case $\alpha =$

0.5. The equation for rents is obtained from indifference of agents to reside in different locations. The equation for wages takes into account this spatial structure of rent by local monopsonistic firms, who try to attract some labor which otherwise will all go to work in business center. Formally, there are two equations:

$$P(R) = w(0) - 2tR - \frac{(w(0) - P(0))}{(1 - 2R/(l_0 V))}; \tag{9}$$

$$w(R) = P(R) + (1 - \frac{2R}{l_0 V})(w(0) - P(R) - 2tR). \tag{10}$$

Substitution of $P(R)$ into the equation for $w(R)$ gives the spatial pattern of wages which repeats the spatial pattern for rents, with some constant shift. Hence, at all points R, the derivatives of both functions are equal: $P'(R) = w'(R)$. It is useful to provide analytical expression to these derivatives, which shown that the lowest absolute value these derivatives are taking at point zero: $|w'|_{min} = 2t + 2(w(0) - P(0))/(l_0 V)$. Thus, equilibrium wage gradient has a nonlinearity increasing with the distance. This is an important observation which can be used to explain the phenomenon of periurbanization. Intuitively, if agents would have a possibility to enter labor markets of local monopsonists, they would influence the movement of wages there making these markets a bit more competitive. But only to some extent. They have a rationale for a marginal travel to work across neighbouring points only if absolute value of spatial gradient is higher critical. A possibility of spatial arbitrage between any points will be considered in the next section. For simplicity, markets for housing are not considered at this stage.

4 Possibilities of Spatial Arbitrage

A set of previous models can be extended by allowing agents to make secondary arbitrage, across any pair of points, using spatial patterns of wages and rents. Excess complexity does not allow to obtain general analytical solution to this problem. In the beginning the model without housing market is considered.

No arbitrage condition across any two points. Another question has to be mentioned: whether this spatial distribution of wages $w(r)$ is consistent with the possibility to do spatial arbitrage across other points different from

the border. In principle, if the model allows for such behaviour (no legal restrictions for employment), some people could be better off, living in point r ad working in r', where $0 < r' < r$. The wage distribution $w^*(r)$ gives the equal utility to all agents, living at any point $r, r < r^*$, and working either at the border, $r = 0$, or at home, r. For any agent, living and working at r' after this wage distribution was established, it is possible to construct a wage profile, which makes him indifferent between staying at r' or moving closer to the border, to some point r:

$$W(r', r) = (w^*(r') - 2p(r' - r))(l_0 - \frac{2|r' - r|}{V})^{\frac{1}{\alpha}-1}. \tag{11}$$

We consider an array of wage profiles, taking r' as a parameter. If $W(r', r) > w^*(r)$ for all $0 < r < r'$, then no additional arbitrage possibilities will emerge, since higher wage is requested for move than those offered. Suppose that for some r' and $r < r'$, $W(r', r) < w^*(r)$. It means that there is an incentive to remigrate[4] to work inside the country B, from r' to r. If all these individual processes influence wages only marginally, they will occur until there will be no "arbitrage opportunities" of this sort. After each period of individual migration decisions the wage distribution near the border will evolve, converging finally to some equilibrium distribution (in the sense of no possibility of increasing utility by working in any other place, including all continuum of possibilities for the continuum of agents, indexed by the point where they live). This equilibrium spatial structure of wages $\Phi(r)$ should satisfy the following properties:
a) $\Phi(r)$ is a continuous decreasing function,
b) $\Phi(0) = w(0)$,
c) $\frac{d\Phi}{dr'} = \frac{\partial W(r,r')}{\partial r}|_{r=r'}$.

Special case $\alpha = 0.5$, $l_0 = 6$. The reason to have this specification is to simplify algebra ($\alpha = 0.5$) and to calibrate results in metric system, where distance is measured in km, speed in km/h, while agents assume to have 6 hours of leisure during the days when they work. We have

$$W(r, r') = (\Phi(r) - 2t(r - r'))(1 - \frac{r - r'}{3V}). \tag{12}$$

[4]The term "remigration" is used to denote a process of finding job at the point different from residence

The last condition (c) is actually the requirement for every local indifference curve $w(r, r')$, constructed at every point of equilibrium wage profile $\Phi(r)$, to be tangent to this wage profile. It leads to a simple differential equation, which can be easily solved:

$$\frac{d\Phi(r)}{dr} = -2t - \frac{\Phi(r)}{3V}, \tag{13}$$

$$\Phi(r) = -6tV + (w(0) + 6tV)e^{-\frac{r}{3V}}. \tag{14}$$

The derived formulae would give the equilibrium wage distribution after all migration processes, not related to renting or buying an apartment.

5 Other Extensions

These extensions show the typical transitional pattern of agent behaviour and can be applied to border problem (like USA-Mexico, or West-East Europe). Here we stay in a framework of a simple model describing the short run behaviour of agents after new source of well paid jobs emerges, while housing market is not yet in equilibrium. The supply of houses in short run is fixed, since construction takes time.

5.1 Possibility to Subrent a Part of House

The next step is to consider the possibility of renting an apartment near the border in order not to travel every day there. It is assumed that in the beginning there is no market for apartments, because all inhabitants inelastically demand their apartments in order to enjoy utility from living there. Then they start to realize that renting part of their apartment may be profitable because of high demand. Who will demand these apartments more? First of all, workers which live further than at the distance r^*. Because they have the lowest wages and because they can not afford travel. The number of these workers is very high, but the amount of square meters of apartments near the border is quite limited. Also, inhabitants would like to offer only some part of their apartments for rent. Suppose that all inhabitants of country B have the same preferences for consuming square meters of apartments and other goods. Then it is possible to find an equilibrium, which will determined what share of apartments will be offered, what will be the price for it and how many migrants from the distant area of B will arrive to rent them.

They will not arrive all, because the wage in competitive industry will start to increase and will make the marginal potential renter indifferent between moving and staying. Mathematically, we have the following equations:

$$U_0 = w(0)^\gamma S^{1-\gamma} = (w(0) + RS')^\gamma (S - S')^{1-\gamma}, \tag{15}$$

$$U_R = w_R^\gamma S^{1-\gamma} = (w(0) - RS')^\gamma (S')^{1-\gamma}, \tag{16}$$

$$w_R = w_B(1 - \frac{M}{P})^{-\beta}. \tag{17}$$

Here S is the area of living space normally chosen by all the citizen of country B before opening the border. It is assumed that it is their private property, they pay nothing for it. P is the population (labor force), which lives far away from the border and is interested in migration there. N is the number of people, who potentially can offer part of their apartments for rent; they live on the border. It is assumed that every person, living on the border, is willing to offer S' square meters of his apartment for one migrant. The assumption of infinite supply of working places on the border rules out the effect of potential wage decrease due to migration; thus the losses of the owner are only in disutility from living on less square meters. In an equilibrium, these losses are just compensated by the gains from rent. A migrant is also indifferent between having wage $w(0)$ on the border, paying rent from it and enjoying only S' square meters - and staying at home with S square meters, but enjoying less wage. When some outflow of migrants occurs, this wage w_B is given by the last equation of the system, for the case of competitive industry with Cobb-Douglas production function, defined as before. The goal is to find the renting price R and the typical square S', offered for rent, from these equations. The solution is given by the formulae:

$$S' = \frac{\sqrt{a^2 + 8aS + 4S^2} + 2S - a}{6}, \tag{18}$$

$$R = \frac{w(0)}{S'}(\frac{S}{S - S'} - 1), \tag{19}$$

$$a \equiv \frac{w_B}{w(0)} S(1 - \frac{N}{P})^{-\beta}. \tag{20}$$

So, according to this model, some migrants will travel to work (T) and then N will then arrive to rent apartments.

5.2 Dynamics

This parts stays a bit apart of static equilibrium models considered above. It focuses on a simple version of including dynamics related to non-stationarity of housing supply over time.

The middle run is defined here as the time, necessary for the development of housing construction and creating a market for it. The long run, which is out of the scope of this model, is defined as the time when the local firms can reallocate their capital and adjust to the transition, implied by the opening of the border for working on it.

It is assumed, that the capacity of the construction industry is exogeneous and represented by some time path $h(t)$. Initially nobody has savings enough to buy a new house (because it is impossible to sell a house far from the border for the price of construction a new one; they have symbolic prices because of the lack of demand). All the population, interested in migration, starts to save money to buy a new house. It is assumed that some minimal consumption is necessary. So, those who currently have higher wage, will spend less time for these savings. First buyers would be those who live not far from the border, as their wage is higher. Then people from more and more distant regions will start to do it. In this linear model it is not obvious whether the prices for houses will increase or decrease with time. If there would be one center, then the time path for prices is more likely to increase in time.

Some estimations of the price dynamics for buying houses at the border are given by the following model. Assume that to buy a flat on the border is considered to be a good investment for all the citizens, but all of them are facing liquidity constraint. That is why they decide to consume at the minimal level C_0 in order to save money to buy a flat. Let $P_f(t)$ be the price dynamic to buy a flat. Then, given the spatial wage distribution $w(r)$, it is possible to find time t, necessary for saving money to buy a flat for an agent, living at the distance r from the border. As $w(r)$ is decreasing function, it is possible to find inverse function of it (at least, for $r < r^*$). Then, the equation

$$P_f(t) = (w(r) - C_0)t \tag{21}$$

can be considered as the implicit function $r = r(P_f(t), t)$. If the density of population (including internal migrants) at r is $\sigma(r)$ and if $\alpha(t)$ is the velocity of construction of new flats at time t, then demand equals supply, when: $\sigma(r(P_f(t), t))dr/dt = \alpha(t)$. This equation defines an equilibrium price process for housing prices on the border.

6 Heterogeneous Agents

Another direction of extension is to add in explanation of periurbanization by allowing agents to have different wealth. The models described so far, were mostly designed to describe a transitory phenomena of wage adjustment after liberalization of labor mobility in space. They can also be used to model the initial stages of urbanization. Now the main accent will be shifted towards equilibrium pricing of real estate in a city and its neighbourhood.

Consider now the case when housing markets are opened for trade, and agents have different initial endowments of wealth. For simplicity, all houses are assumed to be physically identical (like a typical 2 bedroom flat) and differ only in location. The CBD offers high wage, $w(0)$, and agents choose where to reside. Assume first that agents are heterogeneous with respect to their wealth. They still work in CBD, at wage W, but have some additional wealth not derived from their work, and demand housing according to their wealth. As it is shown in [4], there exists an equilibrium mapping of agents into locations, which gives rise to an equilibrium house pricing $R(r)$, normally with a nonlinear gradient. We name these agents "old" since accumulating wealth normally takes time.

6.1 Equilibrium Mapping of Heterogeneous Agents into Locations

The theoretical problem of mapping of heterogeneous agents into location has a fundamental problem of indeterminacy when agents have both a possibility to choose location and land size [9]. However, when market offers physically identical good (it can be either a piece of land or a flat of fixed size) and there is one commuting center, heterogeneous agents can order themselves in

space, so that richer settle closer to CBD.[5]

The paper [4] considers an infinity of agents with given income distribution $f(w)$, who have Cobb-Douglas preferences like in the previous model and have to select a location for settlement. The following system of first-order differential equations defines the general equilibrium of this model:

$$(P'(R) + 2t)(l_0 - \frac{2R}{V}) = \frac{2}{V}(P(R) + 2tR - w(R)), \tag{22}$$

$$w'(R) = -\frac{\nu(R)}{f(w)} \tag{23}$$

(here b is unit distance transport cost, $P(R)$ - rental price of housing per unit of time at distance R, $\nu(R)$ - spatial density of housing available at distance R ,V, t defined before). The second equation of this system is the law of mass preserving. It has a negative sign because richer agents will always locate closer to the city, as they have higher opportunity cost of leisure. The first equation comes from an infinite sequence of individual optimizations. It defines implicitly a mapping of different agents (marked by their income w) into different locations R. This system of equations together with the border conditions

$$P(R_{max}) = A, \qquad w(R_0) = w_{max}, \tag{24}$$

where A is non-locational (physical) component of house price, determines the equilibrium for this model.

Note that the model determines the city radius R_{max}. If it is exogeneous (like in Monaco), only richer agents are able to settle there. If space around a city has no alternative use, then there is also partition of agents into groups $w < \bar{w}$ (poor) and $w > \bar{w}$ (rich), while R_{max} is determined by the equation: $\int_0^{Rmax} \nu(R)dR = \int_{\bar{w}}^{w_{max}} f(w)dw$.

[5] While the US evidence suggests an opposite stylized fact, which is explained by urban economsts via selection of larger land slot further away from the CBD [2], European evidence suggests opposite pattern. Especially it is pronounced in large agglomerations like Moscow or London, where central location has higher rent than peripheric. Some smaller cities like Barcelona do not exhibit so typical radical pattern of pricing real estate, perhaps because factors other than distance to the center play more important role there.

6.2 An Example of Analytical Solution

The solution to this problem not only exists, but can also be constructed analytically for a big class of differentiable distributions. Consider a uniform density of income: $f(w) = 1/(b-a), w \in (a, b)$, and zero otherwise. Here a is the lowest income, b is the highest, and we have normalization: $\int f(w)dw = 1$. Let $\nu(R) = 2\pi R$ (radial symmetry). Then, from the mass preserving condition $w'(R) = -2\pi R/(b-a)$ and the boundary condition $w(0) = b$ it is possible to reconstruct the mapping:

$$w(R) = b - \frac{\pi R^2}{b-a}. \tag{25}$$

Note that the mapping of agents has a parabolic shape in this case. It is because while cohorts of different income groups are of the same size, intervals of distances $[R, R + dR]$ provide more housing as R increases as concentric circles of the same width contain more area further from the center. For the price function there is a differential equation

$$P' + h(R)P = g(R); \qquad h(R) \equiv \frac{-2}{l_0 V - 2R};$$
$$g(R) \equiv -2t + \frac{4tR - 2w(R)}{l_0 V - 2R}. \tag{26}$$

It has a one-dimensional class (with parameter D) on analytical solutions:

$$P(R) = e^{-\int h(R)dR}[D + \int g(R)e^{\int h(R)dR}dR]. \tag{27}$$

It is possible to show (see the details in Appendix) that integration of differential equation given the following equilibrium rent $P(R)$ as the function of distance:

$$P(R) = \frac{1}{(l_0 V/2 - R)^2}[D - (tl_0 V + b)R + 2tR^2 - \frac{\pi}{3(b-a)}R^3]. \tag{28}$$

Note that $D > 0$ since otherwise $P(0) < 0$. It is also useful to find the derivative of this function, which corresponds to price gradient:

$$P'(R) = \frac{1}{(l_0 V/2 - R)^2}[-\alpha + \beta R - \gamma R^2 + \delta R^3]; \tag{29}$$
$$-\alpha \equiv D - \frac{bl_0 V}{2} - \frac{t}{2}(l_0 V)^2; \qquad \beta \equiv 2l_0 Vt > 0;$$
$$-\gamma \equiv 2t + \frac{\pi}{3(b-a)} + \frac{\pi l_0 V}{2(b-a)}; \qquad \delta \equiv \frac{\pi}{b-a}.$$

Asymptotic analysis. Consider the asymptotics when $l_0 V >> 1$ and $b >> 1$. The first condition means that technology is highly developed so that transport speed is high, and agents have enough leisure to invest in spatial arbitrage. The second condition means that there exist agents rich enough. If $R \to 0$, we get:

$$\lim_{R \to 0} P'(R) = -\frac{t}{2} - \frac{2b}{l_0 V}. \tag{30}$$

While it is intuitively clear that price gradient should be negative, its absolute value is not so obvious. The last formulae shows that the higher is the maximal income, the steeper is price gradient in the neighbourhood of the city center. This is also quite intuitive since more rich agents are bidding for limited quantity of well located housing.[6]

Discussion of the results. The previous example shows how to obtain equilibrium spatial profile of rents for a particular income distribution. It also shows that rent gradient depends on different parameters of the model and can be high or low close to the city center, and also change its behaviour when we move away.

6.3 Let us Add Other Agents

The present extension is in the spirit of the model describing border effect. It is possible to assume that migrants are coming from rural areas with little wealth to the city. We may think that there are also young agents in this model who have no wealth and can rent houses in different locations. They have a choice where to settle and where to work. If rent gradient is high enough, they can prefer to settle further away from CBD and travel to work. Young agents take the spatial rent profile $P(R)$ as given and choose where to rent a house and where to work. For some wealth distribution it may happen that rent gradient declines sharper than transport cost.[7] The problem with this type of agents is that they have no sufficient wealth or potential income to pay rent in the city center and close to it. But they may settle at the

[6] It is useful to mention that price gradient can be unboundedly high, when wealth is not limited. This is shown for power distribution $f(w) = w^{-a}$ in [4].

[7] The intuitive explanation for this effect is that wealthier agents value their leisure at higher value. They prefer to buy apartment at closer location from CBD than to commute a particular extra distance, while poorer agents have an opposite choice.

periphery, where $P(R) = A$ and try to work there. They still may not afford transport cost to the city center, but there is no reason to assume that labor demand is located only in the center. In the spirit of the first model, it is possible to find spatial structure of wages consistent with no possibility of indifference of all the agents to go to work in CBD or to work locally. If at some areas rent gradient was high, wage gradient will also be high. If it happens that $|w'(R) - P'(R)| > t$, then their is a possibility for a poor agent, who has low opportunity cost of leisure to utilize this possibility of spatial arbitrage.

Explanation of periurbanization. The process of periurbanization does take place at particular time periods in particular countries. As time goes, parameter V normally increases, t declines, and b increases. The whole spatial pattern of $P(R)$ and $w(R)$ is changing as the result of these changes. While in the urbanization stage property prices in cities have been still affordable for new coming rural migrants and young people leaving families of their parents, the increasing bid of wealthier agents for the best locations decreases these opportunities. But if there is still an opportunity to settle in the periphery of the city and to work in the city but away from its center, some poorer agents would be bettwer off utilizing this option. The fact that migration out of big Spanish cities (Madrid and Barcelona) to their neighbourhood had been observed in 1980ies, can be explained by a higher fraction of young agents born in city rather than rural migrants.

7 Conclusions

1. We live in an increasingly globalized world where people have a technical possibility to live and work in different spatial points. This links otherwise disjoint markets for housing and labor and makes continuous space a mostly relevant framework for modeling.

2. The phenomenon of periurbanization is an important part of internal migration pattern in more developed economies. The effect of a border with a rich country attracting labor from poorer country is another typical source of low-distance migratory movement. The paper presents a set of theoretical models to explain the phenomena described above. The space is treated continuously, how it is done in urban economics. Besides that, job locations

are also treated as continuous. The complex interplay of spatial patterns of housing rents and wages in space creates situations when agents prefer to commute between their residences and work even when work is available at the same place.

3. If there is only initial endowment but no market for housing, the commuting possibilities of workers smooth spatial wage pattern around centers offering higher wage, like urban business center or border with a richer country. The transitory zone in space can reach distances of order 100 km within which part of agents commute to the place of their work. Agents out of this zone are better off to migrate permanently. This creates a market for housing subrent and pushes the demand for housing construction.

4. The models of the paper are based on assumption about Cobb-Douglas utility of agents where transportation reduces both consumption and leisure. This creates a nonlinear rent gradient around CBD. When both labor and housing markets are fully liberalized, spatial patterns with nonlinear gradients for both wages and housing rents emerge. At some stage of the process, some agents find it profitable to move away from city to suburban ring and to commute from there to a job in non-central urban area. This forms a process of periurbanization which was described in empirical literature.

5. A model with agents heterogeneous in wealth is also considered. Rich agents are assumed to derive only part of their income from job in CBD. If housing structure is physically homogeneous and differ only in location, richer agents prefer to settle closer to the center. Their competition by rent bids for urban housing push the rents at so high level that no housing inside a city becomes affordable for poorer class. But the development of fast communication allows poorer people to settle in suburban rings from where they commute to the place of their job inside this city. The model predicts a spatial structure of rents which can be analytically constructed by solving differential equations for any wealth distribution.

Appendix

Consider the equation $P'(R) + h(R)P = g(R)$. Direct calculations of integrals show that

$$\int h(R)dR = \ln|\frac{l_0V}{2} - R|, \qquad e^{\int h(R)dR} = \frac{l_0V}{2} - R;$$

$$\int g(R)e^{\int h(R)dR} = \int 2t(R - \frac{l_0V}{2})dR + \int (2tR - b - \frac{\pi R^2}{b-a})dR =$$

$$2tR^2 - (tl_0V + b)R - \frac{\pi}{3(b-a)}R^3. \qquad (31)$$

Then,

$$P(R) = \frac{1}{(l_0V/2 - R)^2}[D - (tl_0V + b)R + 2tR^2 - \frac{\pi}{3(b-a)}R^3], \qquad (32)$$

where constant D is determined from the condition $P(R_{max}) = A$.

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