Knowledge Spillovers in Cities: The Creation and Transmission of Knowledge

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Abstract

We analyze knowledge spillovers in a search-theoretic spatial equilibrium framework with workers who are heterogeneous in knowledge type. Knowledge spillovers result from random face-to-face interactions between workers in the city. The outcome of those interactions crucially depends on the combination of the interacting individuals' knowledge types. In contrast to previous work, we explicitly model knowledge spillovers as the interplay of two channels: knowledge transmission (imitation) and knowledge creation (innovation). Our results show that if the role of innovation is sufficiently important, individuals choose an excessively narrow range of partners to interact with, leading to lower than socially optimal creation of new ideas, which results in socially inefficient city sizes.

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1. Introduction

? argues that new communication technologies lead to the "death of distance", rendering cities in their role as meeting points irrelevant because individuals can avoid the cost of urban living and still benefit from information flows through new communication methods. However, empirical observation shows that cities are becoming even more important as the increasing concentration of economic activity in space continues. New communication methods may indeed facilitate the transmission of codifiable information with a stable meaning expressed in a standardized system of symbols, but the exchange of complex uncodifiable knowledge brings along distinct challenges. Only real-life face-to-face interactions give people the opportunity to use all relevant forms of communication techniques at the same time. As these interactions are local, spillovers of complex knowledge are also spatially restricted. **?** show that more knowledge intensive industries exhibit a significantly stronger geographic concentration. They also find evidence for the great importance of cities in the process of creating new ideas. Less than 5 percent of product innovations occur outside from metropolitan areas and more than 45 percent of these innovations come from the four metropolitan areas New York City, Los Angeles, San Francisco or Boston. The challenge of quantifying the extent of knowledge spillovers has been faced by **?**, who find that new patents disproportionately often cite patents that were invented in the same city or region. While the importance of local knowledge spillovers as agglomeration force is empirically well established, the literature on the underlying mechanisms is less developed. This further stresses the importance of a more detailed theoretical understanding of these local spillovers. The objective of this paper is to investigate the effects of two different types of knowledge spillovers (imitation and innovation) on urban productivity and city size. We use a search-theoretic spatial equilibrium framework that allows us to analyze both types of knowledge spillovers from urban face-to-face interactions. First, cities give individuals the opportunity to increase their productivity through the process of knowledge transmission (learning). And second, the process of knowledge creation (innovation) increases the rate of technological change in the city, which raises the productivity of each worker affected by the innovation. The transmission of knowledge can be thought of as the result of workers' observation and imitation of each others' techniques. We assume that this process is facilitated when interacting workers have a similar knowledge background. Knowledge creation results in the form of new ideas from the combination of interacting workers' existing knowledge. We adopt the view of **?** and assume that every interaction among workers, independently of their knowledge background, has the potential to bring about innovations. One major difference between these two types of knowledge spillovers is apparent: Workers benefit individually from the process of imitating other workers. The increased productivity directly leads to higher wages. On the contrary innovations are treated as non-excludable local public good in our model. This assumption can be justified as the contribution to the emergence of innovations is often not directly credited to the inventors and thus not fully compensated. The asymmetry in compensation leads to social inefficiencies in workers' choice of face-to-face meetings and location.

To the best of our knowledge there exists no theoretical model that incorporates both types of knowledge spillovers in an urban context. We use a model economy with two asymmetric locations, the city and the periphery. The city provides people with the opportunity to exchange their knowledge via local face-to-face interactions, whereas the periphery does not. Workers in our model can choose the range of other workers in the city they are willing to interact with in order to exchange information. Since individuals do not consider the impact of these interactions on the rate of technological change in the economy, they only accept a range of matches that is smaller than socially optimal.[1] We also show that the resulting suboptimal extent of knowledge spillovers generally leads to socially inefficient city sizes.

The paper is organized as follows: Section 2 reviews the empirical and theoretical literature on the topic of local knowledge spillovers. In sections 3 we introduce the assumptions of our model environment. Section 4 presents the Steady-State Equilibrium of our model economy. In section 5 the market outcome is compared to the outcome that results from the Social Planner's Problem. We further show the different types of inefficiencies that can emerge in our model. Section 6 summarizes and concludes.

2. Review of the Literature

2.1. Review of the Empirical Literature

Due to the intractable nature of knowledge spillovers through face-to-face interactions, it is very difficult to directly measure their extent and sources. Consequently, empirical evidence on different types of knowledge spillovers and their impact on economic growth is scarce. However, there are a few notable contributions that find evidence on the impact of the local industry mix on innovation and productivity growth. These results do not directly address the

[1]There is a very restricted parameter space that results in a range of matches that is larger than socially optimal. More on that finding can be found in section 5.2.

role of knowledge composition that is discussed in our model. However, since industrial diversity and diversity of knowledge in a city are closely related, we can interpret these findings as a reasonable indication for the role of knowledge background.

Independently of the degree of urban specialization or diversification, the literature agrees on the fact that urban density accelerates the emergence of new ideas. Both ? and ? provide significant results and ? show that doubling urban employment density leads to an increase of patent intensity (i.e. patents per capita) of about 20 percent. This observation indicates that a dense urban environment causes a higher rate of face-to-face interactions and in turn the creation of more innovative output.

Similarly to the creation of knowledge, there is no doubt that urban density positively affects the transmission of knowledge in cities as well. Since denser cities bring about more face-to-face interactions and since knowledge is best transmitted via those real-life face-to-face interactions, the individual productivity should be highest in those urban areas. ? show that per-worker productivity is strongly correlated with urban density and hence explain that proximity facilitates the transmission of knowledge.

While the results on the impact of urban density are unanimous, there is some disagreement in the literature when it comes to the role of specialization and diversity. The existing literature distinguishes between two different views of the world. What ? call the Marshall-Arrow-Romer Model suggests that an increased concentration of a particular industry in a city facilitates the exchange and combination of knowledge between individuals and thus leads to the best innovative outcome. This view relies on the idea that sharing the same knowledge background makes it easier for individuals to communicate specific problems in their field. The Marshall-Arrow-Romer Model implies that those innovative meetings are in particular promoted by cities specialized in one specific industry, because those cities feature more face-to-face interactions between people with a similar knowledge background. Silicon Valley, known for its role as pioneer in computer technology is the most famous example for such a highly specialized and innovative region, as was demonstrated by ?.

In contrast, ? argues that innovations can arise from every face-to-face interaction, independently of the interacting individuals' knowledge background. According to her view the most innovative city is a place where people from all different fields of the economy interact unrestrictedly. Therefore she favors diversified cities with no particular specialization in one industry. ? quote the story of the emergence of the financial industry in New York, where grain and cotton merchants saw the need for national and international financial transactions. It was only that need that gave rise to the invention of the industry of

financial services.
? and ? both find empirical evidence for so called Jacobian spillovers, i.e. diversity and not specialization of economic activities enhance growth in cities.[2]
? use data on employment growth between 1956 and 1987 of large industries in 170 U.S. cities. They find that industry-employment growth is significantly positively related to urban diversity of industries and negatively related to urban specialization of industries. ? employ a more direct approach to measure the connection between innovative output and the composition of economic activities in a city. Using the United States Small Business Administration's Innovation Data Base (SBIDB) they can directly observe innovative activity across cities by looking at the number of product introductions across U.S. cities. Their results coincide with ?, i.e. urban diversity enhances the extent of innovative output. But the literature also provides evidence for the existence of so called Marshall-Arrow-Romer spillovers. In contrast to ? and ?, ? find that Marshall-Arrow-Romer spillovers are prevalent for traditional, whereas Jacobian spillovers are prevalent for young high-tech industries. Thus there exists no definite answer to the question which composition of economic activities is best suited for the creation of new knowledge, but there is support for the hypothesis that knowledge creation is at least not harmed by urban diversity.
Besides the fact that knowledge combined in face-to-face interactions leads to the creation of new ideas and thus to a faster rate of technological change, workers can also use these interactions to learn from each other in order to increase their individual productivity. This process is referred to as the transmission of knowledge. There is a wide range of empirical evidence showing that cities are the places that offer the best learning opportunities for workers. ? show that urban workers increase the wage differential over non-urban workers during the time they work and live in the city. This urban wage premium is not lost even when they move from the city to a rural area, supporting the story of skill acquisition in an urban environment. Once workers leave the city, they keep their skills and therefore continue to earn the same nominal wage in the rural area. ? find that learning opportunities are especially strong in cities with a surpassing level of skills, indicating that the contact between highly educated individuals accelerates the accumulation of human capital. The city promoting the optimal environment for individuals to transmit their knowledge in order to increase their productivity is different from a city promoting the optimal conditions to create new knowledge. Having a different knowledge background might be of no harm (or even an advantage) in creating new ideas but the pure transmis-

[2] Economic growth is assumed to be highly correlated with the emergence of innovations.

sion of knowledge in face-to-face interactions is clearly facilitated if interacting individuals have a related body of knowledge. Empirical justification for this statement comes from **?**. They find that specialization of a city in a particular industry leads to a significantly higher rate of wage growth in that industry. This result can be interpreted as the outcome of better learning opportunities in face-to-face interactions among workers with a similar knowledge background.
All these empirical observations can be summarized by three stylized facts (table 1) about how the creation and transmission of knowledge is affected by the economic composition of industries in a city. We use these empirical findings to make predictions about the outcome of face-to-face interactions that we expect to predominantly happen in cities with a specific composition of knowledge.

[TABLE 1]

2.2. Review of the Theoretical Literature

The existing literature on the theoretical foundations of knowledge spillovers in cities mostly focuses on the impact of workers' density on productivity without further distinguishing the underlying forms of knowledge.
? sets up a model of urban learning from face-to-face interactions in which the number of meetings increases in workers' density. This mechanism leads to the agglomeration of skilled workers in cities and urban wage premia. In an extension to his model, the author introduces the assumption that learning only takes place if workers from the same industry meet which reflects the idea of more efficient knowledge transmission between workers with similar knowledge backgrounds adopted in our model.
? uses a similar approach that analyzes the accumulation of young urban workers' skills, assuming that the density of educated workers positively affects the accumulation of skills. The resulting equilibrium features concentration of young and educated workers in cities as observed in reality.
More recently, **?** provide a spatial equilibrium model that examines costly exchange of ideas as agglomeration force. The intensity of knowledge exchange depends on the time devoted to the search for exchange partners, the density of workers and their skill level. Consequently, the exchange of ideas, average abilities of individuals and wage premia are higher in larger cities.
Our work is most closely related to **?**. Their model of knowledge exchange via face-to-face interactions explicitly analyzes the matching of individuals with heterogeneous knowledge backgrounds. This approach incorporates the competing roles of similarity and diversity of knowledge by assuming that there exists an optimal distance of knowledge types that leads to the most efficient transmission

of knowledge. With the number of meetings increasing in density, the efficiency of learning is higher in more densely populated cities.
While all these models capture the higher rate of knowledge spillovers in cities, they focus on the transmission of knowledge and associated buildup of individual skills. Our model extends these contributions as it is the first to explicitly include the creation of knowledge as a distinct process from the transmission of knowledge. This addition enriches the theoretical analysis because the two types of spillovers differ in their dependence on knowledge similarity and also in compensation.

3. Economic Environment

In this section we present the search-theoretic model of a spatial economy incorporating two different types of knowledge spillovers (creation and transmission of knowledge). The model is related to the work of **?** and **?**, but additionally incorporates the creation of knowledge (also referred to as innovative output or innovation in the following) in the city. The basic idea is the following: Cities provide workers with the opportunity to get into contact via face-to-face meetings. We assume that only a dense urban environment gives individuals the chance to engage in face-to-face interactions, whereas a rural area does not (e.g. because the area is too spacious, meeting points like public squares are not prevalent, etc.). In the city, workers are brought together by a random meeting-technology, where the outcome of knowledge transmission and knowledge creation of each interaction is influenced by the combination of the unobservable knowledge types of the meeting partners. The partners' knowledge type and thus the realization of the intensity is unknown before the meeting, but revealed after a first contact. This framework is adopted from **?**, who uses this environment in the context of stochastic job matchings. In this type of model it is crucial to distinguish between a meeting and a match. Whether a meeting between two workers becomes a match depends on the realized productivity.[3] Meetings with low realizations are canceled after a very first contact because it is worthwhile to wait for a better partner (with a more adequate knowledge type) to be matched with. We further adopt the neoclassical assumption that innovative output is costlessly available to everyone in the city and workers are not fully compensated for their created knowledge. This approach makes innovative output a local public good. Its existence gives rise to social inefficiencies

[3]For the rest of the paper the label "contact" is tantamount to "meeting" and the label "face-to-face-interaction" is tantamount to "match".

since the social benefit of generated innovations exceeds the private benefit. Therefore workers accept only the matches that maximize their expected personal outcome, not taking into account that each accepted match contributes to publicly available innovations in the city.

3.1. Basic structure of the economy

Our model economy is populated by infinitely-lived workers. It consists of two heterogeneous regions: The city and the periphery. Time is continuous and in each point of time workers decide in which region to be located. All the action takes place in the city, whereas the periphery is modeled as simple as possible. In the city, individuals have the possibility to interact face-to-face. Living in a crowded urban environment is associated with economic cost. Pollution, road congestion and high house prices are only a few examples for the burden of urban living. Each worker in the city generates congestion costs of $t > 0$. N denotes the number of individuals living in the city, so the total congestion costs each worker faces upon entering the city are tN.[4] There is no crowding in the periphery, so workers living there do not face any congestion costs.

3.2. Economic Agents

Workers are heterogeneous in their horizontally differentiated background of knowledge. The variety of the economy's knowledge base is displayed by a unit circle, represented in figure 1. The approach of using a unit circle to illustrate the economy's knowledge base is adopted from **?** and was used by **?** and **?** among others. Each worker is endowed with a specific knowledge type k, which is represented by its position on the circle's circumference K.[5] The circumference K can be interpreted as the economy's knowledge space representing all types of knowledge in the economy (e.g. economics, mathematics, physics, etc.). The location $k \in K$ is drawn from a uniform distribution and exogenously assigned to each worker. In figure 1 knowledge type k_A is assigned to worker A, whereas knowledge type k_B is assigned to worker B. The distance of k_A and k_B on the circumference is a measure for the horizontal difference between two types of knowledge. There is no vertical differentiation of knowledge types, i.e. all workers have an equal level of education. Furthermore, position k on the unit circle is only of relevance for workers located in the city and irrelevant for

[4]The results of the model analysis are robust to well-established transformations of the congestion cost function. E.g. the results stay unaltered when we use quadratic congestion costs in N, i.e. tN^2. Thus we focus on the easiest case of linear congestion costs.

[5]In the following, we label an individual with that characteristic as worker k.

workers located in the periphery since only the city facilitates the exchange of knowledge via face-to-face interactions.

[FIGURE 1]

Workers are heterogeneous in knowledge background, but homogeneous in preferences. Flow output (equivalent to flow income) y is spent on a homogeneous consumption good. We discuss the determination of flow output y in section 3.4. Flow utility is linear in y, yielding

$$U = U(y) = y. \tag{1}$$

This implies that maximizing the level of lifetime utility is equivalent to maximizing the level of lifetime income.

3.3. Meeting Technology

The reason for workers to enter the city is the opportunity to increase their productivity by the exchange of knowledge. Before introducing the exact modeling strategy of knowledge spillovers, the emergence of meetings (contacts) in the city is clarified. We apply the framework of stochastic job matching used in **?** and assume that there exists a well-behaved meeting function, which gives the number of contacts as a function of the number of workers searching for face-to-face interactions in the city. By using this framework, we are able to generate a connection between the density of a city and the number of face-to-face meetings taking place. Suppose the city is populated by N individuals. A fraction $m \in (0,1)$ of those N individuals is matched (i.e. currently has a face-to-face interaction). We denote the number of matched individuals as M. Thus the fraction of individuals unmatched is $u = 1 - m$ and we denote the number of unmatched individuals as U, which implies that $N = M + U$. It is important to be clear about the difference between a meeting (contact) and a match (face-to-face interaction). Whether a meeting turns into a match is the decision of the individuals who meet and depends on the potential productivity gains. We do not allow for matched workers to search for new partners, so that only unmatched individuals are engaged in the search process. A meeting always requires two parties, one in the first and one in the second position. In **?** the number of job contacts per unit of time depends on the number of firms in the first position and the number of unemployed in the second position. In our modeling framework there are no firms and no unemployed. The number of job contacts is replaced by the number of meetings as well as firms and unemployed are replaced by the number of unmatched workers in the city. Since individuals meet symmetrically, all unmatched workers can either be in the first or second position, thus the meeting function can be described by

$$C = q(U, U). \tag{2}$$

The number of meetings per unit of time is denoted by C. Following **?** it is assumed that the meeting function q is increasing and concave in both arguments and homogeneous of degree $\gamma > 1$. The last assumption ensures that the probability of a meeting increases with the density of unmatched workers in the city. Furthermore q is assumed to fulfill the Inada conditions. Figure 2 illustrates the behavior of the meeting function.

[FIGURE 2]

The meeting technology randomly selects unmatched workers from the set of possible meeting partners U. The meeting rate (the rate at which an unmatched worker has contacts per unit of time) is given by

$$\mu(U) = \frac{C}{U} = \frac{q(U, U)}{U}. \tag{3}$$

Using the assumption of homogeneity of degree $\gamma > 1$ we can rewrite the meeting rate $\mu(U)$ as

$$\mu(U) = \frac{q(U, U)}{U} = \frac{U^{\gamma} q(1, 1)}{U} = U^{\gamma - 1} q(1, 1). \tag{4}$$

In order to derive a meeting rate which is linear in the number of unmatched workers, we set $\gamma = 2$.[6] The expression $q(1, 1)$ determines how many contacts an individual is able to have per unit of time and will thus be denoted as meeting intensity α in the following.[7] Therefore the meeting rate of an unmatched worker in the city can be written as

$$\mu(U) = q(1, 1)U = \alpha U. \tag{5}$$

3.4. Production Technology

We keep production in the periphery as simple as possible in order to focus on the production process in the city. Therefore we assume that the flow output of

[6] This simplifying assumption is also used by **?**.

[7] A value of $\alpha = 0.1$ indicates that during one period of time an unmatched agent can meet 10 percent of all unmatched individuals in the city. A value of $\alpha = 2$ means that during the same time an unmatched individual can talk to each unmatched individual twice.

each individual living in the periphery is equal to the constant $\bar{y}$. For further simplification we set $\bar{y} = 0$.
All structure is put on the production technology in the city. Suppose that an individual of knowledge type k is currently matched with an individual of knowledge type k'. The flow output of worker k crucially depends on the partner's type k' and is represented by

$$y(k, k') = A + e(k, k'). \tag{6}$$

The first expression A denotes the urban total factor productivity (TFP).[8] The TFP is common to all individuals in the city. The second expression $e(k, k')$ denotes the personal effectiveness of individual k currently matched with individual k'. If individual k is currently unmatched, it has a personal effectiveness of zero and flow production is solely determined by the TFP A. Both, the TFP A and the personal effectiveness $e(k, k')$, are influenced by knowledge spillovers resulting from face-to-face interactions in the city.

3.5. Knowledge Spillovers

In this section we answer the question how the extent of knowledge spillovers depends on the combination of knowledge types and we clarify the difference between knowledge transmission and knowledge creation.

3.5.1. Knowledge Transmission

The transmission of knowledge is equivalent to the intellectual exchange described in **?**. The heterogeneity of workers in terms of their position on the unit circle plays a crucial role for the personal effectiveness. Consider two workers: one endowed with knowledge type $k \in K$, the other endowed with knowledge type $k' \in K$. They are brought together by the random meeting technology and suppose, the two workers both accept to be matched. The matching partners' personal effectiveness depends on the distance of their knowledge types k and k' in the knowledge space K, measured by the Euclidean metric $d(k, k')$[9] We assume that the highest degree of knowledge transmission and thus the highest personal effectiveness is attained when the two meeting partners are endowed with exactly the same knowledge type ($k = k'$). In this case it is straightforward to communicate and exchange information. They already use the same

[8] For tractability we assume that total factor productivity is a flow value.

[9] Distance in the knowledge space is just used as a measure for diversity of knowledge background and has nothing to do with physical distance.

vocabulary and techniques, so they can start exchanging knowledge and applying the gained knowledge to attain a higher personal effectiveness right away. The assumption of decreasing knowledge transmission with increasing diversity of knowledge types is justified by the stylized empirical facts from section 2.1 about the outcome of face-to-face interactions in cities. For a better illustration one can think of two economists working in the field of urban economics. Both use the same terminology and the same techniques and once they are matched (i.e. have a research collaboration), they can immediately start to combine their information and become more productive. As the knowledge distance $d(k, k')$ increases, the transmission of knowledge becomes more cumbersome. Since workers do not have a lot in common, they will have problems understanding each others' technical terminology and will not be able to just imitate each others' techniques. Thus, individuals with very heterogeneous knowledge backgrounds have to put in a lot of effort before the transmission of knowledge can begin and once able to communicate they will find it difficult to apply the gained knowledge in their respective fields. Here one can think of a match between an economist and a dentist. Both will have major problems in understanding the vocabulary and imitating each others' techniques. Once they have managed to communicate, it remains questionable whether they can apply the gained knowledge in their respective occupation. The personal effectiveness $e(k, k')$ can be described through the relationship:

$$e(k, k') = e_0 - e_1 d(k, k') \tag{7}$$

The parameter $e_0 > 0$ describes the highest possible level of personal effectiveness that can be attained. This value is achieved when the two meeting partner have exactly the same knowledge background ($k = k'$). The parameter $e_1 > 0$ describes the sensitivity of knowledge transmission to heterogeneity of knowledge types.[10] [11]

[10] $e_1 = 0$ indicates that heterogeneity of knowledge types among individuals is irrelevant for the transmission of knowledge and thus for the personal effectiveness. Each match between two workers, independently of the knowledge types they are endowed with, generates the same amount of knowledge transmission. $e_1 \to \infty$ in turn indicates that only workers with the same knowledge background have a chance to increase productivity through the transmission of knowledge. As soon as the knowledge types differ to a minimal extent they are not able to communicate.

[11] This assumption is in contrast to **?**. They assume that there exists an optimal distance $d^* > 0$ between knowledge types that creates the highest possible productivity. Their justification for this assumption is that "when individuals are too alike, they cannot accomplish much and little knowledge will be obtained". We do not adopt this assumption because we distinguish between two different types of knowledge spillovers, the transmission and creation of knowledge, whereas **?** combine these spillovers into one effect. Once the transmission and creation of knowledge are analyzed separately, it makes sense to assume a maximum effectiveness when agents are alike ($k = k'$), because the pure transmission of knowledge (in the

3.5.2. Knowledge Creation

Matches in the city not only facilitate the transmission of existing knowledge but also lead to the creation of new knowledge that in turn raises the urban level of technology. How does the distance between knowledge types $d(k, k')$ affect the creation of new knowledge? For the creation of knowledge it is not so clear that the outcome should decrease in the distance $d(k, k')$. We can revisit the two examples from above. Two economists working in the same field can form a research collaboration that makes both of them more productive. Additionally they can use the research collaboration to write papers that contribute to the creation of new knowledge. An economist and a dentist will find it very hard to apply the gained information to increase their individual productivity. The used techniques are too different to apply them in their respective occupation and thus we expect the extent of knowledge transmission to be rather low. But it is possible that the combination of their knowledge leads to the creation of new knowledge. For example they could find a way to create a more cost-efficient treatment plan or accounting system. The impact of the knowledge distance $d(k, k')$ on the creation of knowledge is thus less clear than the impact on the transmission of knowledge. Therefore, we are content with the weak assumption that each match, independently of the diversity of knowledge types k and k', creates new knowledge that contributes to the level of technology in the city. This assumption also goes in line with the stylized empirical facts on the extent of knowledge creation in cities. Accordingly, the creation of new knowledge $a(k, k')$ by a currently matched agent with knowledge type k is independent of his partner's knowledge type k' and always equal to $a_0 > 0$, which can be described by

$$a(k, k') = a_0. \tag{8}$$

We adopt the Neoclassical view and assume that created knowledge is a local public good for all individuals living in the city. In the following we make the simplifying assumption that created knowledge is directly translated into the total factor productivity A in the city. In each point of time the TFP is equal to the created knowledge of a matched worker (a_0) times the number of matched individuals in the city (M). Since the created knowledge is assumed to be a local public good, it is equally distributed across all individuals living in the

absence of knowledge creation) is easiest when agents do not have to overcome any knowledge barrier.

city (N), regardless of whether they are currently matched or not. Therefore the relationship can be denoted by

$$A = \frac{Ma_0}{N}. \tag{9}$$

The existence of a second type of knowledge spillover (i.e. the creation of new knowledge) is a potential source for social inefficiencies. Workers choose the range of acceptable matches in order to maximize their personal effectiveness. What they do not take into account is their impact on the creation of knowledge. Section 4 provides an extensive discussion of these inefficiencies. Figure 3 summarizes the impact of the knowledge distance $d(k,k')$ on the extent of knowledge transmission and knowledge creation in urban face-to-face interactions. In figure 3, a_0 is chosen to be larger than e_0. However, since the relation between a_0 and e_0 is an empirical question, our model allows for all other potential coherence between a_0 and e_0.

[FIGURE 3]

3.6. Choice of the Knowledge Spread

In section 3.3 we introduced the meeting technology that determines the rate of contacts an unmatched worker has during one unit of time. We derived the meeting rate $\mu(U) = \alpha U$, which shows that the number of contacts increases linearly at rate α in the density of unmatched individuals U in the city. So what determines which meeting turns into a match and which meeting is canceled after a first contact? Suppose we have a meeting between two individuals endowed with knowledge types k and k'. Given the position k on the unit circle, each individual chooses a knowledge spread $\delta_k > 0$, which determines the range of workers individual k is able to have a face-to-face interaction with. The knowledge spread δ_k is geometrically represented by the arc around knowledge type k leading to a knowledge horizon $[k - \delta_k/2, k + \delta_k/2]$. The knowledge horizon can be interpreted as the set of disciplines, individual k has at least elementary knowledge about. This knowledge is indispensable for individual k to exchange knowledge in face-to-face interactions with an individual of knowledge type k'. Only if the knowledge type k' is located within the knowledge horizon of k, i.e. $k' \in [k - \delta_k/2, k + \delta_k/2]$, a face-to-face interaction is possible. Thus extending the knowledge horizon by increasing the knowledge spread δ_k means that individual k interacts with a wider range of workers in the city.

The unit circle represented in figure 4 displays the knowledge space K. Since worker A with knowledge type k_A is located within the knowledge horizon of individual k, a face-to-face interaction is possible. Individual B with knowledge type k_B is situated outside the knowledge horizon of individual k. Consequently no transmission and creation of knowledge can occur, since worker k has no elementary understanding of B's field of knowledge.

[FIGURE 4]

As workers are ex-ante symmetric[12], we only have to consider symmetric equilibria which implies that all workers choose the same knowledge spread ($\delta_k = \delta \ \ \forall k$). In this case either both individuals accept or both reject to be matched.[13] In the next section we clarify how the choice of δ comes about in the market solution.

3.7. Expected Lifetime Utility

Now, we can determine the expected lifetime utility of a worker being currently matched (V_m) and the expected lifetime utility of a currently unmatched worker (V_u). The value of being matched depends on the knowledge types of individuals currently having a face-to-face interaction and on the number of unmatched workers in the city U. Thus we have $V_m = V_m(k, k', U)$. Since an unmatched individual of knowledge type k has currently no face-to-face interaction, its expected lifetime utility only depends on its own knowledge type k and and the number of unmatched individuals U, yielding $V_u = V_u(k, U)$.
There exists an exogenous separation rate $\lambda > 0$, at which ongoing face-to-face interactions are split up. This separation rate results from idiosyncratic shocks that arrive to interactions at rate λ. Furthermore, there exists a perfect capital market, in which assets can be traded over time at the exogenous interest rate $r > 0$.

The value of a match between workers k and k', given a number of unmatched individuals U, satisfies the following Bellman Equation:

$$rV_m(k, k', U) = \underbrace{A + e(k, k')}_{y(k,k')} + \lambda\Big[V_u(k, U) - V_m(k, k', U)\Big] \tag{10}$$

The value of being matched with worker k' can be interpreted as an asset held by worker k. Given perfect capital markets, the left hand side (capital cost of the

[12] Knowledge types are revealed only after two workers have met, i.e. workers are heterogeneous ex-post, but indistinguishable ex-ante.
[13] If worker k is not located within the knowledge horizon of k', then worker k' is automatically not located within the knowledge horizon of k.

asset) has to equal the right hand side (rate of return on the asset). A currently matched worker of knowledge type k has flow income $y(k, k') = A + e(k, k')$ as long as he stays matched with worker k'. The state changes from matched to unmatched at the exogenous separation rate λ with an associated net return of $V_u(k, U) - V_m(k, k', U)$.

The value of worker k being unmatched at a given number of unmatched individuals U in the city, satisfies the following Bellman Equation:

$$rV_u(k, U) = A + \alpha U \int_{k-\frac{\delta}{2}}^{k+\frac{\delta}{2}} \Big[V_m(k, k', U) - V_u(k, U)\Big] dk' \tag{11}$$

As before the value of being unmatched can be interpreted as an asset held by worker k and with perfect capital markets the left hand side equals the right hand side. The first term on the right hand side is the flow of total factor productivity in the city. The meeting rate $\mu(U) = \alpha U$ gives the number of contacts an individual has during one unit of time. Only individuals with a knowledge type $k' \in [k - \delta/2, k + \delta/2]$ are accepted for a face-to-face interaction.[14] The term inside the integral can be interpreted as the net return from changing the state from being unmatched to being matched with a worker of knowledge type k'.

There are two opposing effects to take into account when worker k chooses the knowledge spread δ. First, increasing the knowledge spread δ extends the knowledge horizon. Thus worker k has face-to-face interactions with a wider range of individuals in the city which increases the probability of turning a meeting into a match. On the other hand, increasing the knowledge spread δ also decreases the expected quality of the partners, k accepts for face-to-face interactions, i.e. individual k also enters matches with individuals that have not much in common with him. This diminishes the transmission of knowledge as individual k may be stuck with a bad match.[15] The opportunity cost of a bad match is the possibility of interacting with more adequate partners in the meantime.

Now it is possible to compute the rate of face-to-face interactions per unit of time. We know that a fraction δ of all meeting partners are accepted for a face-to-face interaction. Thus, the number of matches $p(U)$ per unit of time is

[14] Other partners are not accepted, because worker k is not able to exchange knowledge with them.

[15] It is a bad match from the perspective of worker k. It might be a good match for the economy (or city) as a whole, since new knowledge is created.

given by

$$p(U) = \mu(U)\delta = \alpha\delta U, \tag{12}$$

where $\alpha\delta$ can be interpreted as the matching intensity. The matching rate linearly increases in the density of unmatched individuals U. When workers decide where to locate they use the value of being unmatched (V_u) as their benchmark.[16] Using the fact that the distribution of knowledge types on the unit circle is uniform yields that the expected distance from a matched partner's knowledge type is $\delta/4$. Therefore the value of being unmatched V_u can be written as

$$V_u(\delta, U) = \frac{A}{r} + \frac{\alpha\delta U}{r + \lambda + \alpha\delta U}\frac{e_0 - e_1\frac{\delta}{4}}{r}. \tag{13}$$

The first term on the right hand side (A/r) is the lifetime income generated from the total factor productivity (TFP). Since A is a local public good, worker k can make use of it independently of being currently matched or not. The second part of the second term on the righthand side can be interpreted as the expected income premium an individual gets if it is consistently matched compared to being consistently unmatched. Since individual k is not matched all the time this expected income premium is discounted. The discount rate (first part of the second term) can be interpreted as discounted matching rate.[17]

4. Equilibrium Analysis

In this section, we establish the symmetric Steady State Nash Equilibrium. The equilibrium is defined by the workers' choice of knowledge spread δ, the number of unmatched individuals in the city U and the resulting city size N.

4.1. Steady State Population

The Steady State Equilibrium requires the number of matched and unmatched workers in the city to be constant over time. In the symmetric case, this relationship implies that the flows into and out of the pool of unmatched workers have to equal.

$$\underbrace{\alpha\delta U}_{\substack{\text{flow out of the pool of}\\ \text{unmatched individuals}}} = \underbrace{\lambda M = \lambda(N - U)}_{\substack{\text{flow into the pool of}\\ \text{unmatched individuals}}} \tag{14}$$

[16]This is the state in which they enter the city.
[17]It is the matching rate $\frac{M}{N}$ with the interest rate r in the denominator.

Using this identity, we can derive the number of unmatched individuals in the city as an implicit function of the total city population N in steady state:[18]

$$U = \frac{\lambda}{\lambda + \alpha\delta U} N \tag{15}$$

4.2. Steady State Equilibrium

Workers choose the optimal knowledge spread δ^* by maximizing expected lifetime utility in the city. The optimal knowledge spread δ^* is determined by the trade-off between increasing the probability of having face-to-face interactions and increasing the expected extent of knowledge transmission during a match. Once the optimal knowledge spread δ^* is chosen, individuals move to the city until the levels of lifetime utility in the city and the periphery are equalized and no incentive for moving between the two locations exists (Spatial Equilibrium). Together with equation (15) the equilibrium values N^* and U^* are determined. The Steady State Equilibrium, which we will also refer to as Market Solution in the remainder of the text, is defined as follows:

Definition 1: Steady State Equilibrium
The Steady State Equilibrium is an allocation $\{\delta^*, U^*, N^*\}$ that satisfies the following conditions:

(1) Workers maximize expected lifetime utility by choosing their knowledge spread δ: $\delta^* = \text{argmax}_\delta V_u(\delta, U^*)$.

(2) The level of lifetime utility in the city equals the level of lifetime utility in the periphery: $V_u(\delta^*, U^*) - tN^* = 0$.

(3) The condition for Steady State population is satisfied: $U^* = \frac{\lambda}{\lambda + \alpha\delta^* U^*} N^*$.[19]

Workers do not consider the impact of their choice of knowledge spread δ on the steady state population of unmatched individuals, which gives rise to an inefficiency, that we will refer to as matching externality in the following.
Furthermore, workers do not consider their impact on the local flow of innovation, instead they choose their individual knowledge spread δ to maximize their expected lifetime utility from personal effectiveness. We will refer to this inefficiency as innovation externality.[20]

[18]This expression is analogous to the Beveridge Curve in Labor Market Theory.(**?**)

[19]Conditions (1) and (2) determine the allocation $\{\delta^*, U^*\}$, condition (3) automatically determines N^*.

[20]Both, the matching and innovation externality are discussed extensively in section 5.

Consequently, their choice of δ satisfies the following first order condition:

$$\frac{\partial V_u}{\partial \delta} = \delta^2 + \frac{2(r+\lambda)}{\alpha U}\left(\delta - \frac{2e_0}{e_1}\right) = 0 \tag{16}$$

This condition implicitly defines the equilibrium knowledge spread as a function $\delta(U)$. In the following, we will refer to this condition as the knowledge spread condition KS. We see from KS, that the larger the number of unmatched individuals in the city U, the lower the choice of the knowledge spread δ. Intuitively, this follows from the positive impact of population density on the matching rate, which allows workers to be more picky regarding their interaction partners. This relationship leads to a downward sloping KS-locus in figure 5.

In spatial equilibrium, the levels of lifetime utility have to equalize across locations, such that incentives for relocating disappear. In the context of our model, this definition implies that workers move to the city until the attainable expected lifetime utility in the city $(V_u - tN)$ equals lifetime utility in the periphery (0). Therefore the number of unmatched workers living in the city is determined by the following condition:

$$\underbrace{\underbrace{\frac{\alpha\delta U}{\lambda + \alpha\delta U}\frac{a_0}{r}}_{A/r} \; + \; \frac{\alpha\delta U}{r+\lambda+\alpha\delta U}\frac{e_0 - e_1\frac{\delta}{4}}{r}}_{V_u(\delta,U)} \; - \; tN = 0^{21} \tag{17}$$

By making use of the steady state population condition, the equilibrium number of unmatched individuals in the city as a function $U(\delta)$ is defined. In the following, we will refer to this relation as the equilibrium entry condition EE.

The influence of the knowledge spread δ on equilibrium population N is twofold. An increase in δ raises the matching rate, but also diminishes the average extent of knowledge transmission. This interaction between knowledge spread δ and equilibrium population N leads to a hump shaped form of the EE-locus in figure 5.[22]

[FIGURE 5]

A Steady State Equilibrium emerges for the values of knowledge spread and

[22]Figure 5 shows a hump shaped relationship between δ and U. This coherence automatically implies a hump shaped connection between δ and N.

unmatched workers $\{\delta^*; U^*\}$ in the intersection point of the KS- and the EE-locus as depicted in figure 5. Given the values of δ^* and U^*, equation (15) determines the value of N^*. Using the arbitrarily chosen parameter values in the example above, we derive $\delta^* = 0.265$, which means that in Steady State individuals living in the city accept to have face-to-face interactions with 26.5 percent of meeting partners. The size of the city is $N^* = 7.516$ with $U^* = 3.409$ individuals being unmatched.

5. Social Inefficiencies

5.1. Social Planner's Solution

The previously discussed Steady State Equilibrium gives rise to various inefficiencies. In the following, we will explore the extent and the interactions of those externalities in more detail to assert the social inefficiency of our model economy. In order to do so, we will compare the equilibrium conditions with the optimal choice of a social planner who chooses the knowledge spread δ and the population allocation N simultaneously.
Our model contains three sources of externalities: Congestion externalities arise because individuals do not consider the impact of their location decision on congestion costs. Entering the city bears costs $t > 0$ for every worker living in the city. Matching externalities arise because individuals do not consider the impact of their choice of the knowledge spread δ on the mass of unmatched workers. Most prominently in our model, innovation externalities arise, because individuals do not consider the impact of their choice of the knowledge spread on innovations and thus on the TFP A in the city. The Definition of the Social Planner's solution to our model is as follows:

Definition 2: Social Planner's Solution
The Social Planner's Solution is an allocation $\{\hat{\delta}, \hat{U}, \hat{N}\}$ that satisfies the following conditions:

(1) The Social Planner chooses the knowledge spread and the number of unmatched workers in the city simultaneously in order to maximize the expected lifetime utility of a worker living in the city: $\{\hat{\delta}, \hat{U}\} \in \text{argmax}_{\delta,U} V_u(\delta, U)$.

(2) The condition for Steady State population is satisfied: $\hat{U} = \dfrac{\lambda}{\lambda + \alpha\hat{\delta}\hat{U}}\hat{N}$.

The exact optimality conditions are stated in the appendix. Here, we focus on the effects the Social Planner takes into account when choosing the knowledge spread and the population allocation. First, the optimality condition for the knowledge spread:

$$\frac{\partial V_u}{\partial \delta} = \underbrace{\frac{\partial A}{\partial \delta}}_{\substack{\text{innovation extern.} \\ (>0)}} + \underbrace{\frac{\partial A}{\partial U}\frac{\partial U}{\partial \delta}}_{\substack{\text{matching extern.} \\ (<0)}} + \frac{\partial e}{\partial \delta} + \underbrace{\frac{\partial e}{\partial U}\frac{\partial U}{\partial \delta}}_{\substack{\text{matching extern.} \\ (>0)}} = 0 \tag{18}$$

The matching externalities lead to an equilibrium knowledge spread that is larger than socially optimal. Individuals are too broad in their acceptance of partners, because they do not consider that they prevent potentially more productive matches by lowering the mass of unmatched workers.

The innovation externality leads to an equilibrium knowledge spread that is smaller than socially optimal. The knowledge spread is too narrow as individuals do not consider the innovative output of interactions with diverse knowledge types. Depending on the relative importance of these externalities, it is possible that workers are either too picky or too generous in their choice of partners. In the following analysis, we will focus on the case of a sufficiently important role of innovations such that the innovation externality outweighs the matching externalities and the equilibrium knowledge spread is smaller than socially optimal.

The impact of the inefficiencies on location decisions can be elaborated by the planner's optimality condition for population allocation: The social planner chooses U such that net lifetime utility for the representative worker in the city is maximized, implying the first order condition:

$$\frac{\partial V_u}{\partial U} = \frac{\partial A}{\partial U} + \frac{\partial e}{\partial U} - t\frac{\partial N}{\partial U} = 0 \tag{19}$$

For the case of a smaller than socially optimal equilibrium knowledge spread, the inefficiencies in equilibrium location decisions are again twofold: First, the congestion externality leads to a larger than socially optimal city size as workers do not consider their impact on city-wide congestion costs. Second, the inefficient choice of δ leads to diminished agglomeration forces as knowledge spillovers do not reach their optimal extent. This inefficiency leads to smaller than socially optimal cities.

In summary, we can say that equilibrium choices of δ and N are generally inefficient. The direction and extent of the inefficiencies depend on the relative importance of innovation for the choice of δ and the importance of overall knowledge spillovers for the choice of N.

5.2. Existence of inefficiency patterns

Depending on the parameter configuration, we find three distinct patterns of inefficiency in workers' choices of knowledge spread and location in the market equilibrium relative to the Social Planner's solution: Overselectivity and Underpopulation, Overselectivity and Overpopulation as well as Underselectivity and Overpopulation.
In the following, we verify the existence of these inefficiency patterns by construction.

Case 1 (Overselectivity and Underpopulation):
The first of these cases is marked by excessively narrow knowledge spreads and too small cities in the market equilibrium. Overselectivity can be explained by the relatively large importance of innovation, which workers do not take into account. Underpopulation directly follows from the significant overselectivity in this case, which does not allow the agglomeration force of innovation to develop its full extent. An example for the resulting market equilibrium and social planner allocations are depicted in figure 6 below.

[FIGURE 6]

Case 2 (Overselectivity and Overpopulation):
The case of overselectivity and overpopulation in the market equilibrium is similarly driven by an innovation externality. However, in this case, overselectivity is not as pronounced and therefore the interplay of matching and congestion externality leads to overpopulation as illustrated below.

[FIGURE 7]

Case 3 (Underselectivity and Overpopulation):
In the third case of our equilibrium analysis, underselectivity and overpopulation are prevalent in the absence of innovation ($a_0 = 0$). As the innovation externality is zero in this case, the matching externality leads to chosen knowledge spreads that are larger than socially optimal. The interplay with congestion externalities again leads to overpopulation.

[FIGURE 8]

In summary, these results highlight the relevance of the agglomeration force of innovation in our model. If the role of innovation is sufficiently important,

we find that workers are too picky in the choice of their interaction partners. Depending on the magnitude of this externality, cities can be smaller or larger than socially optimal.
If innovation is irrelevant however, we find excessively large knowledge spreads due to the matching externality and cities that are larger than socially optimal.

5.3. Predicted Inefficiency Pattern

While the preceding section established the existence of three distinct patterns of inefficiency, we will now determine which of these patterns is the most empirically relevant. In order to do so, we calibrate our model's key parameters from existing empirical work. Our goal is not to exactly quantify the effects, but rather to establish the model's qualitative predictions.
The key parameters of our model are the ones that govern the relative importance of knowledge creation and transmission, i.e. e_0, e_1 and a_0. It is impossible to measure the parameters directly as the effect of knowledge transmission on productivity is not directly observable. However, observed urban wage premia can be used as an approximation to their respective impact. We interpret measurements of static urban wage premia as an approximation for the role of the technology level and measurements of dynamic wage premia as an approximation for the role of learning. This interpretation follows the logic, that workers immediately benefit from the higher technology level upon moving to the city, while the buildup of know-how happens over time. For the qualitative predictions of our model, it is the relative importance of innovation and learning that is crucial. Recent studies from **?**, **?**, **?** all find that the static wage premium is more pronounced than the dynamic wage premium. **?** quantify the share of the static premium in urban lifetime earnings premia as 2/3 while the remaining 1/3 stems from dynamic premia. Controlling for observable and unobservable individual characteristics, they find the static premium for Norwegian urban workers to be 3.3%. Accordingly, we set our parameter a_0 to 0.033. The value of the dynamic premium from learning is consequently set to 0.0165. This value corresponds to $e_0 - e_1\delta/2$ in our model. We normalize $e_0 - e_1 1/2 = 0$. As δ is an endogenous variable, we cannot uniquely determine e_0 and e_1, a reasonable parameter configuration in line with the previous findings is to set $e_0 = 0.02$ and $e_1 = 0.04$.
The qualitative predictions of the model are less sensitive to the remaining parameters as long as their magnitude is broadly in line with reality. The parameters of the arrival and separation rate of matches in our model are chosen in line with the literature on matching in the labor market. Following **?**, we set the arrival rate $\alpha = 0.3$ and the separation rate $\lambda = 0.05$. Interest rate r is set to

0.01 and congestion costs $t = 0.1$.
For this parameter configuration, the resulting equilibrium pattern as depicted in figure 9 is marked by overpopulation and overselectivity. Thus our model predicts that workers choose a range of interaction partners that is too narrow and that city size is larger than socially optimal. This qualitative pattern is robust to any parameter configuration that is in line with the existing literature on urban wage premia and matching rates.

[FIGURE 9]

6. Conclusion

There is strong empirical evidence showing the continuing importance of urban knowledge spillovers. **?** show that more than 96 percent of product innovations stem from metropolitan areas and **?** find that the urban wage premium is most accurately explained by learning opportunities in cities.
The aim of this paper is to develop a spatial model that explicitly incorporates two different types of knowledge spillovers (the creation and transmission of knowledge) in cities and to show how they affect the migration decision of individuals. We use the framework of stochastic job matching from **?** and apply it in the context of urban face-to-face interactions. Our model economy incorporates two asymmetric locations: The city and the periphery, where only the city provides individuals with the opportunity to exchange knowledge via face-to-face interactions. In each point of time, workers decide where to be located. Furthermore, workers in the city decide over the range of individuals they are willing to interact with. The intensities of knowledge creation and knowledge transmission in those interactions depend on the similarity of knowledge background of the interacting individuals: First, the individual buildup of skills through knowledge transmission increases in the similarity of knowledge backgrounds. And second, the creation of knowledge is independent of knowledge backgrounds.
The market solution exhibits three sources of inefficiencies. Since created knowledge is a local public good in the city, workers only focus on the buildup of their personal skills when deciding about the range of individuals they accept to be matched with (innovation externality). Congestion externalities arise because individuals do not consider the impact of their location decision on city-wide congestion costs. Matching externalities arise because individuals do not consider the impact of their choice of the knowledge spread on the mass of unmatched individuals.
Depending on the parameter values, we find that workers choose a range of

matching partners that can be smaller or larger than socially optimal. The more important the role of knowledge creation in face-to-face interactions, the more likely it is, that the chosen range of interaction partners is smaller than socially optimal. This means that people overestimate the importance of interacting with other individuals having a relatively similar knowledge background. The interplay of agglomeration and dispersion forces determines the allocation of people in spatial equilibrium. Moving to the city provides the chance to benefit from knowledge spillovers in face-to-face interactions. However, these face-to-face interactions come at the price of urban congestion costs. The model analysis shows that the inefficient decision on the range of individuals to interact with also leads to socially inefficient city sizes. Depending on the chosen parameter values the model's equilibrium city size can be smaller or larger than socially optimal.

For parameter values based on the existing empirical literature on urban wage premia and matching, we find that workers are too picky in their choice of partners and the resulting city size is larger than socially optimal.

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AppendixA. Optimality Conditions for the Social Planner

The optimality conditions for the knowledge spread $\frac{\partial V_u}{\partial \delta} = 0$ and for the number of unmatched agents in the city $\frac{\partial V_u}{\partial U} = 0$ can be expressed as follows:

Optimality condition for the knowledge spread:

$$
\begin{aligned}
\frac{\partial V_u}{\partial \delta} = & \underbrace{\frac{\lambda \alpha U}{(\lambda + \alpha \delta U)^2} \frac{a_0}{r}}_{\frac{\partial A}{\partial \delta}} + \\
& + \underbrace{\frac{\lambda \alpha \delta}{(\lambda + \alpha \delta U)^2} \frac{a_0}{r}}_{\frac{\partial A}{\partial U}} \underbrace{\left(- \frac{\alpha U^2}{\lambda + 2\alpha \delta U} \right)}_{\frac{\partial U}{\partial \delta}} + \\
& + \underbrace{\frac{(r+\lambda)\alpha U (e_0 - e_1 \frac{\delta}{2}) - (\alpha \delta U)^2 \frac{e_1}{4}}{r(r+\lambda+\alpha \delta U)^2}}_{\frac{\partial e}{\partial \delta}} + \\
& + \underbrace{\frac{(r+\lambda)\alpha \delta (e_0 - e_1 \frac{\delta}{4})}{r(r+\lambda+\alpha \delta U)^2}}_{\frac{\partial e}{\partial U}} \underbrace{\left(- \frac{\alpha U^2}{\lambda + 2\alpha \delta U} \right)}_{\frac{\partial U}{\partial \delta}} = 0
\end{aligned}
$$

Optimality condition for the number of unmatched in the city:

$$
\begin{aligned}
\frac{\partial V_u}{\partial U} = & \underbrace{\frac{\lambda \alpha \delta}{(\lambda + \alpha \delta U)^2} \frac{a_0}{r}}_{\frac{\partial A}{\partial U}} + \\
& + \underbrace{\frac{(r+\lambda)\alpha \delta (e_0 - e_1 \frac{\delta}{4})}{r(r+\lambda+\alpha \delta U)^2}}_{\frac{\partial e}{\partial U}} - \\
& - t \underbrace{\frac{\lambda + 2\alpha \delta U}{\lambda}}_{\frac{\partial N}{\partial U}} = 0
\end{aligned}
$$

References